

CHARACTERIZATION OF THE TANGENT SPACE OF MONOTONE TRANSPORT PLANS IN $\mathbf{R} \times \mathbf{R}$ WITH PRESCRIBED FIRST PROJECTION.

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ABSTRACT. We provide a complete description of the tangent space of the cone of monotone plans in $\mathbf{R} \times \mathbf{R}$ with prescribed first projection. We show that elements of this tangent space are essentially made of two simple building-block types of measures.

1. INTRODUCTION.

The study of continuity equation has shown that an appropriate notion of tangent space in the 2-Wasserstein space $\mathcal{P}_2(\mathbf{R}^d)$, at $\varrho \in \mathcal{P}_2(\mathbf{R}^d)$ is given by:

$$\text{Tan}_\varrho \mathcal{P}_2(\mathbf{R}^d) = \overline{\{\nabla \varphi : \varphi \in C_c(\mathbf{R})\}}^{L^2(\varrho, \mathbf{R}^d)} \quad d \in \mathbf{N}. \quad (1.1)$$

While this notion of tangent space has proven to be a valuable tool in many problems, it has also shown its limit especially when dealing with tangent space at non-regular measures. For instance, when ϱ is equal to the average of n delta masses sitting at n distinct points, $\text{Tan}_\varrho \mathcal{P}_2(\mathbf{R}^d)$ is isomorphic to \mathbf{R}^{nd} , which makes it too small to be of great interest.

In [2] and [6], the authors introduced a broader notion of tangent spaces called geometric tangent cone :

$$\mathbf{Tan}_\varrho \mathcal{P}_2(\mathbf{R}^d) := \overline{\{\gamma \in \mathcal{P}_2(\mathbf{R}^{2d}) : (\pi^1, \pi^1 + \varepsilon \pi^2) \# \gamma \text{ optimal for some } \varepsilon > 0\}}^{\mathbf{W}_\varrho} \quad (1.2)$$

Here, $\mathcal{P}_2(\mathbf{R}^{2d})$ denotes the set of transport plans whose first marginal is ϱ and \mathbf{W}_ϱ , its associated metric; see Definition 2.1. π^1, π^2 denote respectively, the first and second projection maps. It is important to note that $\mathbf{Tan}_\varrho \mathcal{P}_2(\mathbf{R}^d)$ is constructed by using the geodesics with respect to the 2-Wasserstein distance. Even though the two tangent spaces come from different perspectives, it was shown that $\text{Tan}_\varrho \mathcal{P}_2(\mathbf{R}^d)$ is isometrically embedded in $\mathbf{Tan}_\varrho \mathcal{P}_2(\mathbf{R}^d)$. As a consequence, the geometric tangent cone offers a richer supply of elements. Behind the well-behaved nature of $\mathbf{Tan}_\varrho \mathcal{P}_2(\mathbf{R}^d)$ hides the fact that this cone adapts well to the possibility of mass splitting that is precisely missing in (1.1).

In spite of the gain the tangent space concept in (1.2) offers over the former concept in (1.1), it is still far from being satisfactory. In the recent work [10], the authors extended the Lagrangian formulation of the one dimensional pressureless

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Euler equations to the multidimensional case. It turns out that the configuration space is better described by *monotone transport plans*, that is transport plans whose support is a monotone set. This means that relying on geodesics could be too restrictive when it comes to describing the configuration space.

In this work, we propose a slightly different concept of tangent cone from convex analysis point of view : the tangent cone of the cone of monotone transport plans whose first projection is ϱ in the space $\mathcal{P}_\varrho(\mathbf{R}^{2d})$. It is noteworthy to mention that in dimension one, the set of optimal maps coincides with the set of monotone maps. We initiate the study of the tangent cone of this cone of monotone transport plans and give a complete characterization for $d = 1$. In the framework developed in [6], we show that this tangent cone is convex. As a result, elements of the tangent cone are linear combinations of two basic types of measures so that on the diffuse part on ϱ , an element of the tangent cone is induced by a map and, at each atomic point its support can be any closed subsets of \mathbf{R} .

This paper is organized in the following way: section 2 provides some notation, definitions, and general results related to $\mathcal{P}_\varrho(\mathbf{R}^2)$ and its associated metric. In Section 3, we give some convergence results and state the main theorem of the paper. In Section 4, the tangent space of the monotone transport plans is proven to be a convex cone in $\mathcal{P}_\varrho(\mathbf{R}^{2d})$. Section 5 deals with the complete description of the tangent space of monotone plans in $\mathcal{P}_\varrho(\mathbf{R}^{2d})$.

2. NOTATION AND DEFINITIONS.

In this section we introduce some notations and recall some standard definitions and state some main results from previous work.

- $\mathcal{P}(\mathbf{R})$ is the set of all Borel probabilities on \mathbf{R} .
- $\mathcal{P}_p(\mathbf{R})$ ($1 \leq p < \infty$) is the set of all probability measures $\mu \in \mathcal{P}(\mathbf{R})$ such that

$$\int_{\mathbf{R}} |x|^p d\mu < \infty.$$

- Given μ and $\nu \in \mathcal{P}_p(\mathbf{R})$, we denote by $\Gamma(\mu, \nu)$ the set of all Borel measures on $\mathbf{R} \times \mathbf{R}$ whose first and second marginal are respectively μ and ν .
- We say that a Borel map \mathcal{T} *pushes* μ *forward* to ν and write $\mathcal{T}\#\mu = \nu$ if $\nu(A) = \mu(\mathcal{T}^{-1}(A))$ for any Borel subset A of \mathbf{R} .
- Let μ, ν be two Borel measures of finite p -moments. Then, the p -th Wasserstein distance between μ and ν is defined by

$$W_p^p(\mu, \nu) = \min \left\{ \int_{\mathbf{R}^d \times \mathbf{R}^d} |x - y|^p d\gamma : \gamma \in \Gamma(\mu, \nu) \right\}. \quad (2.1)$$

The set of minimizers in (2.1) is denoted by $\Gamma_0(\mu, \nu)$.

- A subset $\mathcal{S} \subset \mathbf{R} \times \mathbf{R}$ is monotone if $\langle x_1 - x_2, y_1 - y_2 \rangle \geq 0$ for any $(x_1, y_1), (x_2, y_2) \in \mathcal{S}$.

Definition 2.1 (Distance on $\mathcal{P}_\varrho(\mathbf{R})$). Given $\varrho \in \mathcal{P}_2(\mathbf{R})$, we define

$$\mathcal{P}_\varrho(\mathbf{R}^2) := \left\{ \gamma \in \mathcal{P}_2(\mathbf{R}^2) : \pi^1 \# \gamma = \varrho \right\}.$$

The space $\mathcal{P}_\varrho(\mathbf{R})$ has a structure that looks like the one of Hilbert spaces and is endowed with a distance in the following way : For any $\gamma^1, \gamma^2 \in \mathcal{P}_\varrho(\mathbf{R}^2)$ with the

disintegration $\gamma^k(dx, dy) =: \gamma_x^k(dy) \varrho(dx)$ with $k = 1..2$, the distance $W_\varrho(\gamma^1, \gamma^2)$ is defined by

$$W_\varrho(\gamma^1, \gamma^2)^2 := \int_{\mathbf{R}^d} W(\gamma_x^1, \gamma_x^2)^2 \varrho(dx).$$

Definition 2.2 (Transport Plans). Let $\varrho \in \mathcal{P}_2(\mathbf{R})$ be given.

(i.) *Admissible Plans.* For any $\gamma^1, \gamma^2 \in \mathcal{P}_\varrho(\mathbf{R}^2)$ we define

$$\text{ADM}(\gamma^1, \gamma^2) := \left\{ \alpha \in \mathcal{P}(\mathbf{R}^3) : (\pi^1, \pi^2) \# \alpha = \gamma^1, (\pi^1, \pi^3) \# \alpha = \gamma^2 \right\}.$$

(ii.) *Optimal Plans.* For any $\gamma^1, \gamma^2 \in \mathcal{P}_\varrho(\mathbf{R}^2)$ we define

$$\text{OPT}(\gamma^1, \gamma^2) := \left\{ \alpha \in \text{ADM}(\gamma^1, \gamma^2) : \right. \\ \left. W_\varrho(\gamma^1, \gamma^2)^2 = \int_{\mathbf{R}^{3d}} |y^1 - y^2|^2 \alpha(dx, dy^1, dy^2) \right\}.$$

In the following theorem, we collect some fundamental results obtained in [6] about the distance in $\mathcal{P}_\varrho(\mathbf{R})$.

Theorem 2.3. Let $\varrho \in \mathcal{P}_2(\mathbf{R})$ be given.

(i.) The function W_ϱ is a distance on $\mathcal{P}_\varrho(\mathbf{R}^2)$ and lower semicontinuous with respect to the weak convergence in $\mathcal{P}(\mathbf{R}^2)$. We have

$$W_\varrho(\gamma^1, \gamma^2)^2 = \min_{\alpha \in \text{ADM}(\gamma^1, \gamma^2)} \int_{\mathbf{R}^3} |y^1 - y^2|^2 \alpha(dx, dy^1, dy^2)$$

for all $\gamma^1, \gamma^2 \in \mathcal{P}_\varrho(\mathbf{R}^2)$, and thus $\text{OPT}(\gamma^1, \gamma^2)$ is nonempty.

(ii.) The set $(\mathcal{P}_\varrho(\mathbf{R}^2), W_\varrho)$ is a complete metric space.

The space $\mathcal{P}_\varrho(\mathbf{R})$ enjoys some vector space like properties when endowed with the following algebraic structure; see[6].

Definition 2.4 (Operations in $\mathcal{P}_\varrho(\mathbf{R}^2)$). Let $\varrho \in \mathcal{P}(\mathbf{R})$ be given.

(i.) *Scalar multiplication.* For any $\gamma \in \mathcal{P}_\varrho(\mathbf{R}^2)$ and $s \in \mathbf{R}$ let

$$s\gamma := (\pi^1, s\pi^2) \# \gamma \in \mathcal{P}_\varrho(\mathbf{R}^2).$$

(ii.) *Addition.* For any pair $\gamma^1, \gamma^2 \in \mathcal{P}_\varrho(\mathbf{R})$ and $\beta \in \text{ADM}(\gamma^1, \gamma^2)$, we define the *addition* with respect to β by

$$\gamma^1 +^\beta \gamma^2 = (\pi^1, \pi^2 + \pi^3) \# \beta.$$

We denote the collection of all *additions* as β varies in $\text{ADM}(\gamma^1, \gamma^2)$ by

$$\gamma^1 \oplus \gamma^2 := \left\{ (\pi^1, \pi^2 + \pi^3) \# \beta : \beta \in \text{ADM}(\gamma^1, \gamma^2) \right\} \subset \mathcal{P}_\varrho(\mathbf{R}).$$

Note that if either of γ^i is induced by a map, then $\gamma^1 \oplus \gamma^2$ reduces to a singleton.

We introduce the following subset of $\mathcal{P}_\varrho(\mathbf{R}^2)$:

$$C_\varrho := \left\{ \gamma \in \mathcal{P}_\varrho(\mathbf{R}^2) : \text{spt } \gamma \text{ is a monotone subset of } \mathbf{R} \times \mathbf{R} \right\}. \quad (2.2)$$

It is shown in [10] that C_ϱ is a closed convex cone with respect to the algebraic structure and the distance above. We recall the following theorem in [6] the proof of which uses the completeness of the metric space $(\mathcal{P}_\varrho(\mathbf{R}^2), W_\varrho)$.

Proposition 2.5 (Metric Projection). *Let $\varrho \in \mathcal{P}_2(\mathbf{R}^d)$ be given and $C_\varrho \subset \mathcal{P}_\varrho(\mathbf{R}^{2d})$ a closed. For any $\gamma \in \mathcal{P}_\varrho(\mathbf{R}^{2d})$ there is a unique $\mathcal{P}_{C_\varrho}(\gamma) \in C_\varrho$ with*

$$W_\varrho(\gamma, \mathcal{P}_{C_\varrho}(\gamma)) \leq W_\varrho(\gamma, \eta) \quad \text{for all } \eta \in C_\varrho.$$

3. PRELIMINARY.

In this section, we consider the scaling map $\mathbf{S}^\lambda : x \mapsto \lambda x$ and the translation map $\mathbf{T}^u : x \mapsto x - u$ where $u \in \mathbf{R}$ and $\lambda \in \mathbf{R}$. We state standard results on the actions of these maps on the Wasserstein distance.

Lemma 3.1. *Let $\mu, \nu \in \mathcal{P}_2(\mathbf{R})$ and assume $\lambda > 0$. Then*

$$W_2(\mathbf{S}^\lambda \# \mu, \mathbf{S}^\lambda \# \nu) = \lambda W_2(\mu, \nu) \quad \text{and} \quad W_2(\mathbf{T}^u \# \mu, \mathbf{T}^u \# \nu) = W_2(\mu, \nu)$$

Proof. Let $\gamma \in \Gamma_0(\mu, \nu)$. Note that $(\mathbf{S}^\lambda, \mathbf{S}^\lambda) \# \gamma \in \Gamma(\mathbf{S}^\lambda \# \mu, \mathbf{S}^\lambda \# \nu)$. Thus,

$$\int |x - y|^2 d(\mathbf{S}^\lambda, \mathbf{S}^\lambda) \# \gamma = \int |\mathbf{S}^\lambda(x) - \mathbf{S}^\lambda(y)|^2 d\gamma = \int |\lambda x - \lambda y|^2 d\gamma = \lambda^2 W_2^2(\mu, \nu)$$

Thus, as $(\mathbf{S}^\lambda, \mathbf{S}^\lambda) \# \gamma \in \Gamma(\mathbf{S}^\lambda \# \mu, \mathbf{S}^\lambda \# \nu)$,

$$W_2^2(\mathbf{S}^\lambda \# \mu, \mathbf{S}^\lambda \# \nu) \leq \lambda^2 W_2^2(\mu, \nu) \quad (3.1)$$

By Symmetry, we have

$$W_2^2(\mu, \nu) \leq \frac{1}{\lambda^2} W_2^2(\mathbf{S}^\lambda \# \mu, \mathbf{S}^\lambda \# \nu) \quad (3.2)$$

We combine (3.1) and (3.2) to obtain the first equality. We establish the second equality in a similar manner. \square

Lemma 3.2. *For $\lambda > 0$, $\gamma_1, \gamma_2 \in \mathcal{P}_\varrho(\mathbf{R})$, we have*

$$W_\varrho((\pi^1, \lambda(\pi^2 - \pi^1)) \# \gamma_1, (\pi^1, \lambda(\pi^2 - \pi^1)) \# \gamma_2) = \lambda W_\varrho(\gamma_1, \gamma_2)$$

Proof. Note that if $\gamma \in \mathcal{P}_\varrho(\mathbf{R})$ and $\bar{\gamma} = (\pi^1, \lambda(\pi^2 - \pi^1)) \# \gamma$ then we have the following disintegration:

$$\bar{\gamma} = \int \bar{\gamma}_x d\varrho \quad \text{with} \quad \bar{\gamma}_x = \lambda(\text{id} - x) \# \gamma_x$$

for ϱ almost every x . In other words,

$$\bar{\gamma}_x = \mathbf{S}^\lambda \circ \mathbf{T}^x \# \gamma_x. \quad (3.3)$$

We use lemma 3.1 above to compute

$$W^2(\mathbf{S}^\lambda \circ \mathbf{T}^{1x} \# \gamma_{1x}, \mathbf{S}^\lambda \circ \mathbf{T}^{2x} \# \gamma_{2x}) = \lambda^2 W^2(\gamma_{1x}, \gamma_{2x}).$$

This, in view of (3.3), yields

$$W_\varrho^2((\pi^1, \lambda(\pi^2 - \pi^1)) \# \gamma_1, (\pi^1, \lambda(\pi^2 - \pi^1)) \# \gamma_2) = \lambda^2 W_\varrho^2(\gamma_1, \gamma_2)$$

\square

Proposition 3.3. *Let $\lambda > 0$, $\mu \in \mathcal{P}_\varrho(\mathbf{R}^2)$. Set $\gamma_\lambda := (\pi^1, \pi^1 + \lambda\pi^2) \# \mu$ and denote by $\mathcal{P}_{C_\varrho}(\gamma_\lambda)$ its metric projection onto C_ϱ as provided by Proposition 2.5. If $\mathbf{m} \in C_\varrho$ then*

$$W_\varrho \left(\left(\pi^1, \frac{\pi^2 - \pi^1}{\lambda} \right) \# \mathcal{P}(\gamma_\lambda), \mu \right) \leq W_\varrho \left(\left(\pi^1, \frac{\pi^2 - \pi^1}{\lambda} \right) \# \mathbf{m}, \mu \right) \quad (3.4)$$

In particular, for $\{\mathbf{m}_\lambda\}_\lambda \in C_\varrho$

$$\left(\pi^1, \frac{\pi^2 - \pi^1}{\lambda} \right) \# \mathbf{m}_\lambda \xrightarrow{W_\varrho} \mu \implies \left(\pi^1, \frac{\pi^2 - \pi^1}{\lambda} \right) \# \mathcal{P}(\gamma_\lambda) \xrightarrow{W_\varrho} \mu$$

Proof. By definition of the projection,

$$W_\varrho(\mathcal{P}(\gamma_\lambda), \gamma_\lambda) \leq W_\varrho(\mathbf{m}, \gamma_\lambda) \quad (3.5)$$

Applying lemma 3.2 to (3.5) we obtain (3.4). \square

Let $\mu \in \mathcal{P}_\varrho(\mathbf{R}^2)$ and set

$$\Lambda_\mu := \left\{ \tau \in \mathbf{R} : \tau > 0 \text{ and } (\pi^1, \pi^2 + \tau\pi^1) \# \mu \in C_\varrho \right\}. \quad (3.6)$$

The next lemma states that either Λ_μ is empty or is an open interval.

Lemma 3.4. *If $\tau_0 \in \Lambda_\mu$ and $0 < \tau < \tau_0$ then $\tau \in \Lambda_\mu$.*

Proof. Let $\tau, \tau_0 > 0$ and set

$$\gamma_0 = (\pi^1, \pi^2 + \tau_0\pi^1) \# \mu \quad \text{and} \quad \gamma = (\pi^1, \pi^2 + \tau\pi^1) \# \mu$$

Note that, as π^1, π^2 are continuous,

$$\text{supp} \gamma_0 = \overline{(\pi^1, \pi^2 + \tau_0\pi^1) [\text{supp} \mu]}.$$

Let $(A_1, B_1), (A_2, B_2)$ be in the image of $\text{supp} \mu$ by $(\pi^1, \pi^1 + \tau\pi^2)$ and choose $(x_1, y_1), (x_2, y_2) \in \text{supp} \mu$ such that

$$A_1 = x_1, \quad A_2 = x_2, \quad B_1 = x_1 + \tau y_1, \quad B_2 = x_2 + \tau y_2.$$

We set,

$$R := (A_2 - A_1)(B_2 - B_1) = (x_2 - x_1)^2 + \tau(y_2 - y_1)(x_2 - x_1).$$

We claim that $R \geq 0$ provided that $\tau_0 \in \Lambda_\mu$. To see this, assume

$$\tau_0 \in \Lambda_\mu \quad \text{and} \quad (y_2 - y_1)(x_2 - x_1) < 0.$$

Thus,

$$R > (x_2 - x_1)^2 + \tau_0(y_2 - y_1)(x_2 - x_1)$$

Therefore, as $(x_1, x_1 + \tau_0 y_1), (x_2, x_1 + \tau_0 y_2)$ belong to the image of $\text{supp} \mu$ by $(\pi^1, \pi^1 + \tau\pi^2)$, the monotonicity of the support of $(\pi^1, \pi^2 + \tau_0\pi^1) \# \mu$ implies that

$$(A_2 - A_1)(B_2 - B_1) = R > (x_2 - x_1)^2 + \tau_0(y_2 - y_1)(x_2 - x_1) \geq 0 \quad (3.7)$$

Since (3.7) holds for any $(A_1, B_1), (A_2, B_2)$ is in the image of $\text{supp} \mu$ by $(\pi^1, \pi^1 + \tau\pi^2)$, we conclude that the image of $\text{supp} \mu$ by $(\pi^1, \pi^1 + \tau\pi^2)$, is monotone. As a result

$$\text{supp} \gamma = \overline{(\pi^1, \pi^2 + \tau\pi^1) [\text{supp} \mu]}$$

is monotone by a density argument. It follows that $\gamma \in C_\varrho$ and so $\tau \in \Lambda_\mu$.

Lemma 3.5. *Let $\mu \in \mathcal{P}_\varrho(\mathbf{R}^2)$. Then, there exists a sequence $\{\mu_n\}_n \subset \mathcal{P}_\varrho(\mathbf{R}^2)$ such that the support of μ_n is contained in $\mathbf{R} \times [-n, n]$ and,*

$$\mu_n \xrightarrow{W_\varrho} \mu$$

Proof. Write $\mu = \int \mu^x d\rho$ and define $P_n(\xi) := \xi \chi_{|\xi| \leq n}$. Consider a sequence of measures $\{\mu_n\}_n \subset \mathcal{P}_\varrho(\mathbf{R}^2)$ such that $\mu_n^x = P_n \# \mu^x$. Clearly, the support of μ_n is contained in $\mathbf{R} \times [-n, n]$ and

$$W^2(\mu_n^x, \mu^x) \leq \int_{\mathbf{R}} |P_n(\xi) - \xi|^2 d\mu^x = \int_{|\xi| > n} |\xi|^2 d\mu^x$$

As, $x \longrightarrow \int_{|\xi| > n} |\xi|^2 d\mu^x \in L^1(\varrho)$ and converges to 0, by the Lebesgue dominated convergence theorem, $W_\varrho(\mu_n, \mu)$ converges to 0. \square

In the sequel, let us write $\varrho \in \mathcal{P}_2(\mathbf{R}^2)$ as

$$\varrho = \sum_{i=1}^{\infty} m_i \delta_{x_i} + \bar{\varrho}.$$

where $\{x_i\}_{i=1}^{\infty} \subset \mathbf{R}$ is the set of atoms of ϱ and $\bar{\varrho}$ is the diffuse part of ϱ and $m_i \geq 0$ so that

$$\sum_{i=1}^{\infty} m_i + \bar{\varrho}(\mathbf{R}) = 1.$$

Lemma 3.6. *Let $\mu \in \mathcal{P}_\varrho(\mathbf{R}^2)$ such that*

$$\mu = \sum_{i=1}^{\infty} \nu_i \times \delta_{x_i} + (\text{id}, u) \# \bar{\varrho}. \quad (3.8)$$

for some $u \in L^2(\bar{\varrho})$ and Borel measures ν_i on \mathbf{R} . Set

$$\mu_N = \sum_{i=1}^N \nu_i(\xi) \times \delta_{x_i} + \sum_{i=N+1}^{\infty} \delta_0(\xi) \times \delta_{x_i} + (\text{id}, u) \# \bar{\varrho}.$$

Then,

$$\mu_N \xrightarrow{W_\varrho} \mu$$

Proof. Note that

$$W^2(\delta_0, \nu_i) = \int_{\mathbf{R}} |\xi|^2 d\nu_i \quad (3.9)$$

Using (3.8), we obtain that

$$\sum_{i=1}^{\infty} \int_{\mathbf{R}} |\xi|^2 d\nu_i \leq \int_{\mathbf{R}} |\xi|^2 d\mu \quad (3.10)$$

Since

$$W_\varrho^2(\mu_N, \mu) = \sum_{i=N+1}^{\infty} W^2(\delta_0, \mu^{x_i}) = \sum_{i=N+1}^{\infty} W^2(\delta_0, \nu_i). \quad (3.11)$$

As μ is of finite second moment, we combine (3.9)-(3.11), to obtain that

$$\lim_{N \rightarrow \infty} W_\varrho^2(\mu_N, \mu) = 0.$$

\square

In order to state our main result, we introduce the tangent space of the monotone transport plans:

$$\mathbb{T}_{C_\varrho} := \overline{\{\mu \in \mathcal{P}_\varrho(\mathbf{R}^2) : (\pi^1, \pi^1 + \tau\pi^2) \# \mu \in C_\varrho \text{ for some } \tau > 0\}}^{W_\varrho}$$

Theorem 3.7. \mathbb{T}_{C_ϱ} is a closed convex cone with respect to the algebraic structure given in Definition 2.4. Furthermore, $\mu \in \mathbb{T}_{C_\varrho}$ if and only if μ has the following representation :

$$\mu = (\text{id}, g) \# \bar{\varrho} + \sum_{i=1}^{\infty} (\nu_i \times \delta_{y_i}) \quad (3.12)$$

for some $g \in L^2(\bar{\varrho})$, a sequence $\{y_i\}_{i=1}^{\infty} \subset \mathbf{R}$ and some family of Borel probability measures $\{\nu_i\}_{i=1}^{\infty}$ on \mathbf{R} .

The proof of this theorem follows:

4. THE TANGENT CONE OF MONOTONE TRANSPORT PLANS IS CONVEX.

In this section we prove the part of the main theorem that is concerned with the fact the tangent space is a convex cone.

Lemma 4.1. Let $\mu_i, \bar{\mu}_i \in \mathcal{P}_\varrho(\mathbf{R}^2)$ $i = 1, 2$ and $\mu \in \mu^1 \oplus \mu^2$. Then there exists $\bar{\mu} \in \bar{\mu}^1 \oplus \bar{\mu}^2$ such that

$$W_\varrho(\mu, \bar{\mu}) \leq W_\varrho(\bar{\mu}^2, \mu^2) + W_\varrho(\mu^1, \bar{\mu}^1)$$

Proof. we refer the reader to ([6] Proposition 4. 24). \square

Proposition 4.2. (1) \mathbb{T}_{C_ϱ} is a cone.

(2) \mathbb{T}_{C_ϱ} is convex : if $\mu_1, \mu_2 \in \mathbb{T}_{C_\varrho}$ then $\mu^1 \oplus \mu^2 \subset \mathbb{T}_{C_\varrho}$.

Proof. (i) $\lambda \in \mathbf{R}_+$ and $\mu \in \mathbb{T}_{C_\varrho}$. Then, there exist $\{\mu_n\}_{n=1}^{\infty}$ and $\{\tau_n\}_{n=1}^{\infty}$ such that $\{\mu_n\}_{n=1}^{\infty}$ converges to μ with respect to W_ϱ and $\tau_n \in \Lambda_{\mu_n}$. Set $\nu_n = (\pi^1, \lambda\pi^2) \# \mu_n$ and note that

$$\left(\pi^1, \pi^1 + \frac{\tau_n}{\lambda} \pi^2\right) \# \nu_n = (\pi^1, \pi^1 + \tau_n \pi^2) \# \mu_n \in C_\varrho$$

Thus, $\frac{\tau_n}{\lambda} \in \Lambda_{\nu_n}$. On the other hand, using lemma 3.1, we easily show that

$$W_\varrho(\nu_n, \nu) = \lambda W_\varrho(\mu_n, \mu)$$

with $\nu := (\pi^1, \lambda\pi^2) \# \mu$. Thus, $\{\nu_n\}_{n=1}^{\infty}$ converges to ν with respect to W_ϱ . So, $\lambda\mu \in \mathbb{T}_{C_\varrho}$. we conclude that \mathbb{T}_{C_ϱ} is a cone.

(ii) $\mu^i \in \mathbb{T}_{C_\varrho}$ $i = 1, 2$. There exists a positive sequence $\{\tau_n\}_{n=1}^{\infty}$ and $\{\mu_n^i\}_{n=1}^{\infty} \subset \mathcal{P}_\varrho(\mathbf{R}^2)$ $i = 1, 2$ such that $\{\mu_n^i\}_{n=1}^{\infty}$ converges to μ^i with respect to W_ϱ and

$$\gamma_n^i = (\pi^1, \pi^1 + \tau_n \pi^2) \# \mu_n^i \in C_\varrho.$$

Let $\mu \in \mu^1 \oplus \mu^2$. Then, by lemma 4.1, there exists $\mu_n \in \mu_n^1 \oplus \mu_n^2$ such that

$$\mu_n \xrightarrow{W_\varrho} \mu. \quad (4.1)$$

Let $\beta_n \in \text{ADM}(\mu_n^1, \mu_n^2)$ so that

$$\mu_n^1 = (\pi^1, \pi^2) \# \beta_n, \quad \mu_n^2 = (\pi^1, \pi^3) \# \beta_n, \quad \text{and} \quad \mu_n^1 +^{\beta_n} \mu_n^2 = (\pi^1, \pi^2 + \pi^3) \# \beta_n,$$

We use the identities

$$(\pi^1, \pi^1 + \tau_n \pi^2) \circ (\pi^1, \pi^2) = (\pi^1, \pi^1 + \tau_n \pi^2)$$

and

$$(\pi^1, \pi^1 + \tau_n \pi^3) \circ (\pi^1, \pi^2) = (\pi^1, \pi^1 + \tau_n \pi^3)$$

to get respectively

$$\gamma_n^1 = (\pi^1, \pi^1 + \tau_n \pi^2) \# \beta_n \quad \text{and} \quad \gamma_n^2 = (\pi^1, \pi^1 + \tau_n \pi^3) \# \beta_n$$

Note that

$$\begin{aligned} \left(\pi^1, \pi^1 + \frac{\tau_n}{2} \pi^2 \right) \# \mu_n &= \left(\pi^1, \pi^1 + \frac{\tau_n}{2} \pi^2 \right) \# (\pi^1, \pi^2 + \pi^3) \# \beta_n \\ &= \left(\pi^1, \pi^1 + \frac{\tau_n}{2} (\pi^2 + \pi^3) \right) \# \beta_n \end{aligned}$$

That is,

$$\begin{aligned} \left(\pi^1, \pi^1 + \frac{\tau_n}{2} \pi^2 \right) \# \mu_n &= \frac{1}{2} (\pi^1, \pi^1 + \tau_n \pi^2) \# \beta_n + \frac{1}{2} (\pi^1, \pi^1 + \tau_n \pi^3) \# \beta_n \\ &= \frac{1}{2} (\gamma_n^1 + \gamma_n^2) \end{aligned} \tag{4.2}$$

As, $\gamma_n^i \in C_\varrho$ and C_ϱ is convex, (4.2) implies that

$$\left(\pi^1, \pi^1 + \frac{\tau_n}{2} \pi^2 \right) \# \mu_n \in C_\varrho.$$

This combined with (4.1) yields that $\mu \in \mathbb{T}_{C_\varrho}$. And so, $\mu^1 \oplus \mu^2 \subset \mathbb{T}_{C_\varrho}$. \square

5. CHARACTERIZATION OF THE TANGENT SPACE.

In this section, we give a complete characterization of the tangent space of monotone plans. Using the convexity property established in the previous section, we show that elements of the tangent cone of monotone plans are made essentially of two basic components.

We recall

$$\varrho = \sum_{i=1}^{\infty} m_i \delta_{x_i} + \bar{\varrho}.$$

where $\{x_i\}_{i=1}^{\infty} \subset \mathbf{R}$ is the set of atoms of ϱ and $\bar{\varrho}$ is the diffuse part of ϱ and $m_i \geq 0$ so that

$$\sum_{i=1}^{\infty} m_i + \bar{\varrho}(\mathbf{R}) = 1.$$

Lemma 5.1. *Let $\gamma \in C_\varrho$. Then, there exists monotone function u , a countable set $B := \{y_i\}_{i=1}^{\infty} \subset \mathbf{R}$ and a family of Borel probability measures $\{\nu_i\}_{i=1}^{\infty}$ such that*

$$u \in L^2(\bar{\varrho}) \quad \text{and} \quad \gamma = (\text{id}, u) \# \bar{\varrho} + \sum_{i=1}^{\infty} \nu_i \times \delta_{y_i}. \tag{5.1}$$

Proof. Let Γ be a maximal monotone extending set for $spt(\gamma)$ and u the corresponding set valued map. Let A be the set of all points x at which $u(x)$ has more than one point. As $u(x)$ is a closed interval in \mathbf{R} , A has at most countably many points, that is, $A = \{m_i\}_{i=1}^\infty$. Let K be a ϱ measurable set such that $\varrho(K^c) = 0$ and the disintegration of γ with respect to ϱ is uniquely given as $\gamma = \int \gamma_x d\varrho$ with $\{\gamma_x\}_{x \in K}$. Let $x \in A^c$ so that $u(x)$ is a singleton. If, in addition, $x \in K$ then $\gamma_x = \delta_{u(x)}$. Clearly

$$\gamma = \int \delta_{u(x)} d\varrho + \sum_{i=1}^\infty \gamma_i \times \delta_{m_i}$$

for some family of Borel probability measures $\{\gamma_i\}_{i=1}^\infty$. As $\varrho = \sum_{i=1}^\infty m_i \delta_{x_i} + \bar{\varrho}$, we have

$$\gamma = \int \delta_{u(x)} d\bar{\varrho} + \sum_{i=1}^\infty \nu_i \times \delta_{y_i}$$

for some $\{y_i\}_{i=1}^\infty \subset \mathbf{R}$. The fact that γ has a finite second moment ensures that $u \in L^2(\bar{\varrho})$. □

Proposition 5.2. *If $\mu \in \mathbb{T}_{C_\varrho}$ then μ has the following representation :*

$$\mu = (\text{id}, g) \# \bar{\varrho} + \sum_{i=1}^\infty (\nu_i \times \delta_{y_i}) \quad (5.2)$$

for some $g \in L^2(\bar{\varrho})$ and family of Borel probability measures $\{\nu_i\}_{i=1}^\infty$ on \mathbf{R} .

Remark 5.3. We are going to show below that any $g \in L^2(\bar{\varrho})$ and any choice of family of Borel probability measures $\{\nu_i\}_{i=1}^\infty$ will give rise to elements of \mathbb{T}_{C_ϱ} .

Proof. Let $\mu \in \mathbb{T}_{C_\varrho}$. Then, there exists $\{\mu_n\}_{n=1}^\infty \subset \mathcal{P}_\varrho(\mathbf{R}^2)$ converging to μ and a sequence of positive numbers $\{\tau_n\}_{n=1}^\infty$ such that

$$\gamma_n := (\pi^1, \pi^1 + \tau_n \pi^2) \# \mu_n \in C_\varrho.$$

By lemma 5.1, γ_n have the representation (5.1). Since $\mu_n = \left(\pi^1, \frac{\pi^2 - \pi^1}{\tau_n}\right) \# \gamma_n$, μ_n have the same representation as (5.1) without u being necessary monotone. Thus, there exists a family of functions $\{u_n\}_{n=1}^\infty \subset L^2(\bar{\varrho})$, $A_n = \{y_i^n\}_{i=1}^\infty \subset \mathbf{R}$ and a family of probability measures $\{\nu_i^n\}_{i=1}^\infty$ such that

$$\mu_n = \int_{\mathbf{R}} \delta_{u_n(x)} d\bar{\varrho} + \sum_{i=1}^\infty \nu_i^n \times \delta_{y_i^n}.$$

By setting $A = \cup A_n$, we rewrite γ_n as

$$\mu_n = \int_{A^c} \delta_{u_n(x)} d\bar{\varrho} + \sum_{i=1}^\infty \bar{\nu}_i^n \times \delta_{s_i}$$

where $A = \{s_i\}_{i=1}^\infty$ and $\bar{\nu}_j^n = \nu_j^n$ if $s_j = y_j^n$.

Note that

$$W_\varrho^2(\mu_n, \mu_m) = \int_{A^c} |u_n - u_m|^2 d\bar{\varrho} + \sum_{j=1}^\infty W^2(\bar{\nu}_j^n, \bar{\nu}_j^m).$$

As $\{\mu_n\}_{i=1}^\infty$ is Cauchy, using completeness of the $L^2(\bar{\varrho})$ and $\mathcal{P}_2(\mathbf{R})$, we obtain that $\{\mu_n\}_{i=1}^\infty$ converges to

$$\mu = (\text{id}, g) \# \bar{\varrho} + \sum_{i=1}^{\infty} (\nu_i \times \delta_{s_i}) \quad (5.3)$$

for some $g \in L^2(\bar{\varrho})$ and borel measures ν_i on \mathbf{R} . The uniqueness of the limit ensures the result. \square

Lemma 5.4. *Let $g \in L^2(\varrho)$. Then,*

$$\mu := (\text{id}, g) \# \varrho \in \mathbb{T}_{C_\varrho} \quad (5.4)$$

Proof. There exists $\{g_n\}_{n=1}^\infty \subset \mathcal{C}_c^\infty(\mathbf{R})$ such that $\{g_n\}_{n=1}^\infty$ converges to g in $L^2(\varrho)$. Denoting $\mu_{g_n} := (\text{id}, g_n) \# \varrho$, we note that

$$W_\varrho^2(\mu_{g_n}, \mu) = \int_{\mathbf{R}} |g - g_n|^2 d\varrho.$$

Thus, $\{\mu_{g_n}\}_{n=1}^\infty$ converges to μ with respect to W_ϱ . As $\{g_n\}_{n=1}^\infty \subset \mathcal{C}_c^\infty(\mathbf{R})$, we choose a sequence of positive numbers $\{\tau_n\}_{n=1}^\infty$ such that $1 + \tau_n \|g'_n\|_\infty > 0$. Then, we obtain that

$$\gamma_n = (\pi^1, \pi^1 + \tau_n \pi^2) \# \mu_{g_n} \in C_\varrho.$$

We conclude that $\mu \in \mathbb{T}_{C_\varrho}$. \square

Lemma 5.5. *Let x_0 be an atom of ϱ and write $\varrho = m_0 \delta_{x_0} + \tilde{\varrho}$ so that $m_0 + \tilde{\varrho}(\mathbf{R}) = 1$. Set*

$$\mu := m_0 \nu \times \delta_{x_0} + \delta_0 \times \tilde{\varrho} \quad (5.5)$$

where μ is a Borel probability measure with compact support. Then $\mu \in \mathbb{T}_{C_\varrho}$.

Proof. Let $\alpha > 0$ such that $[-\alpha, \alpha]$ is the smallest symmetric interval containing the support of ν . We consider an increasing sequence $\{\alpha_n\}_{n=1}^\infty \subset (0, \alpha)$ such that α_n converges to α . Define

$$S_n(x) := \begin{cases} x & \text{on } (-\infty, x_0 - \frac{\alpha_n}{n}) \cup [x_0 + \frac{\alpha_n}{n}, \infty) \\ x_0 - \frac{\alpha_n}{n} & \text{on } (x_0 - \frac{\alpha_n}{n}, x_0) \\ x_0 + \frac{\alpha_n}{n} & \text{on } [x_0, x_0 + \frac{\alpha_n}{n}) \end{cases}$$

and

$$P_n(\xi) := \begin{cases} x_0 - \frac{\alpha_n}{n} & \text{on } (-\infty, -\alpha_n) \\ x_0 + \frac{\xi}{n} & \text{on } [-\alpha_n, \alpha_n] \\ x_0 + \frac{\alpha_n}{n} & \text{on } (\alpha_n, \infty). \end{cases}$$

We note that S_n is monotone increasing with exactly one jump at x_0 of gap the interval $[x_0 - \frac{\alpha_n}{n}, x_0 + \frac{\alpha_n}{n}]$. We use P_n, S_n to construct the following measure :

$$\mathbf{m}_n := m_0 P_n \# \nu(d\xi) \times \delta_{x_0}(dx) + (\text{id}, S_n) \# \tilde{\varrho}(dx).$$

As S_n is monotone and P_n has values in $[x_0 - \frac{\alpha_n}{n}, x_0 + \frac{\alpha_n}{n}]$, it is immediate that the support of \mathbf{m}_n is monotone.

We derive a truncation for μ , by setting

$$\mu_n(dx, d\xi) = \left(\pi^1, + \frac{\pi^2 - \pi^1}{\frac{1}{n}} \right) \# \mathbf{m}_n.$$

That is,

$$\mu_n(dx, d\xi) = \sum_{i=1}^{\infty} m_i \delta_{\bar{S}_n(x_i)}(d\xi) \times \delta_{x_i}(dx) + \bar{P}_n \# \nu(d\xi) \times \delta_{x_i}(dx) + (\text{id}, \bar{S}_n) \# \bar{\rho}(dx).$$

Here,

$$\bar{S}_n = \frac{S_n - \text{id}}{\frac{1}{n}} \quad \text{and} \quad \bar{P}_n = \frac{P_n - \text{id}}{\frac{1}{n}}.$$

A simple computation gives

$$\bar{S}_n(x) := \begin{cases} 0 & \text{on } (-\infty, x_0 - \frac{\alpha_n}{n}) \cup (x_0 + \frac{\alpha_n}{n}, \infty) \\ \frac{x_0 - x}{\frac{1}{n}} - \alpha_n & \text{on } [x_0 - \frac{\alpha_n}{n}, x_0] \\ \frac{x_0 - x}{\frac{1}{n}} + \alpha_n & \text{on } (x_0, x_0 + \frac{\alpha_n}{n}]. \end{cases}$$

and

$$\bar{P}_n(\xi) := \begin{cases} -\alpha_n & \text{on } (-\infty, \alpha_n) \\ \xi & \text{on } [-\alpha_n, \alpha_n] \\ \alpha_n & \text{on } (\alpha_n, \infty). \end{cases}$$

Clearly, as α_n converges to α ,

$$\limsup_{n \rightarrow \infty} W^2(\bar{P}_n \# \nu, \nu) \leq \limsup_{n \rightarrow \infty} \int_{\mathbf{R}} |\bar{P}_n \#(\xi) - \xi|^2 d\nu = 0. \quad (5.6)$$

Observe that

$$|\bar{S}_n(x)| \leq \left| \frac{x_0 - x}{\frac{1}{n}} \right| + \alpha_n \leq 2\alpha \quad (5.7)$$

and

$$\lim_{n \rightarrow \infty} \bar{S}_n(x) = 0 \quad \bar{\varrho} - a.e. \quad (5.8)$$

By using the Lebesgue dominated convergence theorem, (5.7) and (5.8) lead to

$$\lim_{n \rightarrow \infty} \int_{\mathbf{R}} |\bar{S}_n(x)|^2 d\bar{\varrho} = 0. \quad (5.9)$$

Since

$$W_{\varrho}^2(\mu_n, \mu) = \int_{\mathbf{R}} |\bar{S}_n(x)|^2 d\bar{\varrho} + m_0 W^2(\bar{P}_n \# \nu, \nu), \quad (5.10)$$

We use (5.6) and (5.9) to obtain

$$\mu_n \xrightarrow{W_{\varrho}} \mu.$$

□

Now, let us give a proof of the main theorem stated in section 3.

Proof of the main theorem.

The fact that $\mathbb{T}_{C_{\varrho}}$ is a convex cone is established in Proposition 4.2. In view of the results in Proposition 5.2, we only need to show the converse statement. Assume μ has the representation (5.2). By lemma 3.5 and lemma 3.6, it suffices to show that

$$\mu := (\text{id}, g) \# \bar{\varrho} + \sum_{i=1}^N (\nu_i \times \delta_{x_i}) + \sum_{i=N+1}^{\infty} (\delta_0 \times \delta_{x_i}) \quad (5.11)$$

for some positive integer N , $g \in L^2(\bar{\varrho})$ and Borel measures ν_i with support contained in closed and bounded intervals, is an element of \mathbb{T}_{C_ϱ} . Furthermore, as \mathbb{T}_{C_ϱ} is a convex cone, the proof reduces to showing that μ in (5.4) and (5.5) belong to \mathbb{T}_{C_ϱ} . These are obtained respectively in lemma 5.4 and lemma 5.5. \square

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